Asymptotic Behavior of the Hadamard Walk in the Central Limit Theorem of the Open Quantum Random Walk with Time-Dependence

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Abstract
The continuous-time open quantum walk and continuous-time quantum walk have been investigated [C. Pellegrini, Continuous time open quantum random walks and non-Markovian Lindblad master equations, J Stat Phys (2014)154:838-865; O. Muelken, A. Blumen, Continuous-time quantum walks: Models for coherent transport on complex networks, Physics Reports, Volume 502, 37-87 (2011); E. Farhi, S. Gutmann, Quantum computation and decision trees, Phys. Rev. A 58 (1998) 915]. In Chaobin Liu et al. [arXiv:1604.05652v1 [quant-ph] 19 Apr 2016] the continuous-time open quantum walk are introduced as the formulation of quantum dynamical semigroups of trace-preserving and completely positive linear maps (or quantum Markov semigroups) on graphs, and an open question related to the asymptotic behavior of the continuous-time open quantum walk on \( \mathbb{Z} \) in which the swap operator is related to the Hadamard gate was proposed in the conclusions section of the paper. In the present paper we introduce the continuous-time open quantum walk in the Central Limit Theorem and use a discretization process to enable us answer the open question in the conclusions section of Chaobin Liu et al. [arXiv:1604.05652v1 [quant-ph] 19 Apr 2016].

Keywords
Open Quantum Random Walk; Quantum Walk, Dual Process; Limit Theorem; Time-Dependence; Discrete Distribution; Hadamard Gate

I. Introduction
According to Konno [5] the Open Quantum Random Walk (OQRW) was introduced in order to model the quantum efficiency in biological systems and quantum computing and it is based on the non-unitary dynamics induced by the local environments [6, 7]. These random walks deal with density matrices instead of pure states. In [7], Attal et al. developed the quantum trajectory approach for OQRW and by using this concept, they have shown the central limit theorem for OQRW’s on the -dimensional integer space \( \mathbb{Z}^d \) [8].

This paper is organized as follows. In Section 2, we review the OQRW in the central limit theorem (CLT). In Section 3 we introduce time-dependence in the OQRW in the CLT via a certain matrix. In order to study the continuous-time OQRW in the CLT on \( \mathbb{Z} \), we introduce in Section 4, a discretization process which is popular in statistical distribution theory to obtain the discrete analogue of continuous distributions. We also show the asymptotic behavior of the time-dependent Hadamard walk in the CLT of the OQRW, thus giving an alternate solution to the problem posed in the conclusions section of Chaobin Liu et al. [4].

II. Open Quantum Random Walk in the Central Limit Theorem
Let \( M_2 \) be the algebra of \( 2 \times 2 \) matrices. Equip \( M_2 \) with the inner product \( A, B = \text{Tr}(A^* B) \), \( A, B \in M_2 \), and define \( M = \bigotimes_{x \in \mathbb{Z}} M_2 \). Let \( k \in K = (-\pi, \pi] \). For \( a = (a(x))_{x \in \mathbb{Z}} \),

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in the position space and \( f(k) \) in the Fourier domain, define the Fourier transform and the inverse Fourier transform between the position space and the Fourier domain as follows:

\[
\hat{a}(k) = \sum_{x \in \mathbb{Z}} e^{-ikx} a(x), \quad \hat{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} f(k) \, dk
\]

For \( k \in K \), let \( M^{-\otimes_M} \frac{1}{2\pi} \mathbb{R} \). For \( A = \{A(x)\}_{x \in \mathbb{Z}} \in M \), define the Fourier transform between \( M \) and \( M \) as follows:

\[
\hat{A}(k) = \sum_{x \in \mathbb{Z}} e^{-ikx} A(x), \quad \hat{a}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} A(k) \, dk
\]

and similarly for the inverse Fourier transform. Consider \( M_2 \) and define the following matrices: \( L_B(A) = BA \) and \( R_B(A) = AB \), then we see that the dynamics of the OQRW on \( M \) is given by

\[
\rho^{(n+1)} = (L_B A T + L_C R_C T^*) \rho^{(n)}, \quad \text{where} \ T \text{ and } T^* \text{ are translations in the position space of the walker defined as } (Ta)(x) = a(x+1) \text{ and } (T^* a)(x) = a(x-1), \quad \text{where } a = \{a(x)\}_{x \in \mathbb{Z}}.
\]

By induction on \( n \), we can write \( \rho^{(n)} = (L_B R_B T + L_C R_C T^*)^n \rho^{(0)} \).

In the Fourier space the evolution becomes

\[
\rho^{(n+1)} = (L_B A T T^* + L_C R_C T^*)^n \rho^{(0)}, \quad \text{and similarly for the inverse Fourier transform. Recall for the OQRW on } M, \text{ the chirality space of the walker is given by } H_C = \text{span}\{L, R\}, \text{ the position space is given by } H_S = \text{span}\{x : x \in \mathbb{Z}\}, \text{ and the Hilbert space is given by the tensor product } H = H_C \otimes H_S.
\]

Let \( B, C \) be two matrices on \( H \) such that \( B^* B + C^* C = I \). Now we introduce a so-called dual process to the OQRW.

**Definition 1** [5]

The dual process of the OQRW generated by \( B, C \) is the process \( Y_n = \{Y_n(k)\}_{k \in K} \in M \) defined by

\[
Y_n(k) = (e^{jLB}T + e^{-jLB}T^*)^n \rho^{(0)}
\]

**Theorem 2** [5]

The probability distribution of the OQRW is given by

\[
\rho^{(n)} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \rho^{(0)} \, dk, \quad \text{where the initial state is given by} \ \rho^{(0)} = \rho_0 \otimes |0\rangle \langle 0|
\]

Whilst the OQRW is based on the non-unitary dynamics induced by the local environment, observe that the Central Limit Theorem induced by the quantum walk (QW) is obtained (not necessarily) as a special case of Theorem 5.2 [8] as follows

**Corollary 3:**

Consider the stationary open quantum walk random walk on \( \mathbb{Z} \) associated to the operators \( \{B, C\} \), where \( B^* B + C^* C = I \), and \( B + C = \text{Unitary Matrix} \). Assume that the completely positive map

\[
L(\rho) = B\rho B^* + C\rho C^*
\]

admits a unique invariant state \( \rho_\infty \). Let \( \rho_n, X_n \in \mathbb{N} \) be the quantum trajectory process to this open quantum walk, then, \( X_n \Rightarrow N(0, \sigma^2) \) in \( \mathbb{R} \) where

\[
\sigma^2 = Tr \left( C\rho_\infty C^* - B\rho_\infty B^* \right)^2 + 2Tr \left( C\rho_\infty C^* - B\rho_\infty B^* \right)R,
\]

where \( R \) is the solution of the equation \( R - L^* \left( L \right) = C^* C - B^* B \).

On the other hand we also have the following, which is also (not necessarily) a special case of Theorem 5.2 [8]

**Corollary 4:**

Consider the stationary open quantum walk random walk on associated to the operators \( \{B, C\} \), where \( B^* B + C^* C = I \), and \( B + C \neq \text{Unitary Matrix} \). Assume that the completely positive map

\[
L(\rho) = B\rho B^* + C\rho C^*
\]

admits a unique invariant state \( \rho_\infty \). Let \( \rho_n, X_n \in \mathbb{N} \) be the quantum trajectory process to this open quantum walk, then, \( X_n \sim \frac{N(0, \sigma^2)}{\mu} \) in \( \mathbb{R} \), where \( \mu = Tr \left( C\rho_\infty C^* - B\rho_\infty B^* \right) \), and

\[
\sigma^2 = Tr \left( C\rho_\infty C^* - B\rho_\infty B^* \right)^2 - 2\mu Tr \left( C\rho_\infty C^* - B\rho_\infty B^* \right) R - 2\mu Tr(\rho_\infty R),
\]

where is the solution of the equation...
Remark 5: Note that \( \{\rho_n, X_n\}_{n \in \mathbb{N}} \) can be considered a Markov chain, and for details see Konno [5].

III. Open Quantum Random Walk in the Central Limit Theorem with Time-Dependence

In order to introduce time-dependence, we will take 
\[
\begin{align*}
B &= -\frac{i}{\sqrt{2}} \begin{pmatrix}
\cosh(\theta) & 0 \\
0 & \cosh(\theta)
\end{pmatrix} \\
C &= \frac{i}{\sqrt{2}} \begin{pmatrix}
0 & \cosh(\theta) \\
0 & 0
\end{pmatrix}
\end{align*}
\]

Note that the matrix \( B + C \) is a special case of the time-dependent coin in eqn (7) of [9]. Moreover, \( B + C \) gives the Hadamard matrix, 
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\]

With \( B, C \) as defined in this section, observe that the completely positive map \( L(\rho) = B\rho B^* + C\rho C^* \) admits the unique invariant state 
\[
\rho_{\infty} = \frac{1}{2} I_2, \text{ where } I_2 \text{ is the } 2 \times 2 \text{ identity matrix.}
\]

Put 
\[
\rho_{n} = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\]

and since \( L(R) = BRB^* + CRC^* \), then upon solving \( R - L^*(R) = C^*C - B^*B \) with \( B, C \) as defined in this section, we get
\[
R = \begin{pmatrix}
\cosh(2\theta) & e^{-2\theta}\cosh(2\theta) \\
e^{2\theta}\cosh(2\theta) & 1
\end{pmatrix}
\]

Direct computation from Corollary 3 implies that 
\[
\sigma^2 = 3\cosh(2\theta), \text{ and therefore, } X_{n \sqrt{n}} \Rightarrow N\left(0, 3\cosh(2\theta)\right)
\]

IV. Discretization of the Open Quantum Random Walk in the Central Limit Theorem with Time-Dependence

Using discretization criterion to obtain the discrete analogue of continuous distributions is popular in statistical distribution theory, and for a survey of methods the reader should consult [10]. In [11] the discrete analogue of the normal distribution was introduced as follows:

\[
P(Y = k) = \frac{k + 1 - \mu}{\sigma} - \frac{k - \mu}{\sigma}, \quad k = 0, \pm 1, \pm 2 \text{ where } \sigma > 0 \;
; \quad -\infty < \mu < +\infty; \quad \tilde{O}(.) \text{ is the cumulative distribution function of the standard normal distribution. Using this process to discretize, we can write the discrete version of } N\left(0, 3\cosh(2\theta)\right) \text{ as }
\]

\[
P(x, t) = \frac{1}{\sqrt{3\cosh(2\theta)}} \left( \frac{x + 1}{\sqrt{\cosh(2\theta)}} \right) - \frac{x}{\sqrt{\cosh(2\theta)}}, \quad x = 0, \pm 1, \pm 2;
\]

\( \tilde{O}(.) \) is the cumulative distribution function of the standard normal distribution, and \( t \) is the time step. Now we get an alternate solution to the problem in the conclusions of Liu et.al[4] by examining \( \lim_{t \to \infty} \frac{P(x, t)}{\sqrt{t}} \). The graph below shows that in the CLT of the OQRW with time-dependence, the asymptotic behavior is close to normal.

FIG 1: \( \lim_{t \to \infty} \frac{P(x, t)}{\sqrt{t}} \)

References


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